Relativistic bounce-averaged quasilinear diffusion equation for low-frequency electromagnetic fluctuations

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A relativistic bounce-averaged quasilinear diffusion equation is derived to describe stochastic particle transport associated with low-frequency electromagnetic fluctuations in a nonuniform magnetized plasma. Expressions for the relativistic quasilinear diffusion coefficients are calculated explicitly for magnetically-trapped particle distributions in axisymmetric magnetic geometry in terms of drift-bounce resonant contributions associated with low-frequency fluctuations which conserve the first adiabatic invariant.
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I. INTRODUCTION

One of the outstanding problems in magnetospheric physics is the nature of the mechanism(s) responsible for the transport and energization of trapped (radiation-belt) particles during magnetic storms. Relativistic radiation-belt electrons are of special interest because of their damaging effects on spacecraft electronics. During magnetic storms, MeV electron fluxes in Earth's magnetosphere typically drop by a few orders of magnitude early in the storm and then rise to values one to two orders of magnitude above pre-storm values over a period of a day or two.[1] Recent observational studies have shown strong correlations between these flux variations and the occurrence of hydromagnetic fluctuations in the 2 to 10 mHz ULF frequency range.[2,3] In related work, simulation results of Elkington, Hudson, and Chan[4] suggest that the ULF fluctuations may play an important role in the radial transport and energization of the MeV electrons through magnetic-drift wave-particle resonances. More generally, it has long been established that trapped magnetospheric particles can exhibit stochastic behavior in the presence of multiple wave-particle resonances[5–7] leading to dissipationless transport.

A. General quasilinear theory

Stochastic transport of charged particles interacting with low-frequency electromagnetic waves can faithfully be modeled by a quasilinear diffusion equation in which wave-particle resonances are explicitly taken into account. Quasilinear theory was originally developed to describe the collisionless evolution of an unstable velocity-space particle distribution in a uniform plasma.[8,9] The derivation of a quasilinear kinetic equation for a nonuniform magnetized plasma in the full phase space using action-angle coordinates was first performed by Kaufman.[10,11] This formulation, as is shown below, explicitly displays the following three wave-particle resonances: the gyroresonance (in which the wave frequency ω is equal to harmonics of the cyclotron frequency ω_c), the bounce-resonance (in which the wave frequency ω is equal to harmonics of the bounce frequency ω_b), and the drift-resonance (in which the wave frequency ω is equal to harmonics of the magnetic-drift frequency ω_d).

The derivation of a relativistic quasilinear Vlasov equation describing collisionless diffusion in action space begins with the standard relativistic Vlasov equation written in Hamiltonian form as

$$\frac{\partial F}{\partial t} + \{F,H\} = 0, \tag{1}$$

where F is the Vlasov distribution function, H is the single-particle relativistic Hamiltonian, and $\{,\}$ is the Poisson bracket. The Vlasov equation (1) can be written in terms of the *action-angle* coordinates (J_i, θ^i) (with $i=1,2,3$), with which the Poisson bracket has the form

$$\{F,H\} \equiv \frac{\partial F}{\partial \boldsymbol{\theta}} \cdot \frac{\partial H}{\partial \mathbf{J}} - \frac{\partial F}{\partial \mathbf{J}} \cdot \frac{\partial H}{\partial \boldsymbol{\theta}}.$$

Next, we expand the Vlasov distribution F and the Hamiltonian H in powers of a small dimensionless parameter ϵ associated with wave-perturbation amplitude:

$$\begin{aligned} F &= F_0(\mathbf{J};\tau) + \epsilon\, \delta F(\mathbf{J},\boldsymbol{\theta},t), \\ H &= H_0(\mathbf{J}) + \epsilon\, \delta H(\mathbf{J},\boldsymbol{\theta},t), \end{aligned} \tag{2}$$

where F_0 and H_0 are functions of the action variables $\mathbf{J}$, such that $\{F_0,H_0\} \equiv 0$, and F_0 is also a slow function of time ($\tau \equiv \epsilon^2 t$), i.e., $\partial F_0/\partial t \equiv \mathcal{O}(\epsilon^2)$. The first-order perturbation terms δF and δH are associated with the presence of waves perturbing the background medium whose typical wave time scales are short compared to the quasilinear evolution time scale of the background medium.

Upon expanding the Vlasov equation (1) in powers of ϵ and averaging over the fast-wave time scales, we obtain the following equations for F_0 and δF:

$$\frac{\partial F_0}{\partial \tau} = -\overline{\{\delta F, \delta H\}}, \tag{3}$$

$$\frac{\partial \delta F}{\partial t} + \{\delta F, H_0\} = -\{F_0, \delta H\}, \tag{4}$$

where $\overline{(\cdots)}$ denotes averaging with respect to the fast-angle variables and time, and we have omitted the nonlinear term $\{\delta F, \delta H\}$ in the evolution equation for δF (as required by the quasilinear analysis). We now consider the Fourier decomposition

$$\begin{pmatrix} \delta F \\ \delta H \end{pmatrix} = \sum_\kappa \sum_{\mathbf{m}} \begin{pmatrix} \delta \tilde{F}_{\mathbf{m}}(\mathbf{J};\omega_\kappa) \\ \delta \tilde{H}_{\mathbf{m}}(\mathbf{J};\omega_\kappa) \end{pmatrix} \exp i(m^j \theta_j - \omega_\kappa t), \tag{5}$$

where $\mathbf{m} \equiv (m^1, m^2, m^3)$ labels Fourier components (each index m^i takes integer values from $-\infty$ to $+\infty$), the sum Σ_κ denotes a sum over normal modes with eigenfrequencies ω_κ, and $\delta\tilde{F}_{-\mathbf{m}}(-\omega_\kappa) \equiv [\delta \tilde{F}_{\mathbf{m}}(\omega_\kappa)]^*$ as follows from the reality condition on the perturbation fields. Substituting this decomposition into the right side of Eq. (3) yields[12]

$$\begin{aligned} -\overline{\{\delta F, \delta H\}} &= \frac{\partial}{\partial \mathbf{J}} \cdot \overline{\left(\delta F \frac{\partial \delta H}{\partial \boldsymbol{\theta}} \right)} \\ &= \frac{\partial}{\partial \mathbf{J}} \cdot \left(\sum_{\mathbf{m},\kappa} i \mathbf{m} \delta \tilde{F}^*_{\mathbf{m}}(\mathbf{J};\omega_\kappa) \delta \tilde{H}_{\mathbf{m}}(\mathbf{J};\omega_\kappa) \right) \\ &\equiv \frac{\partial}{\partial \mathbf{J}} \cdot \left[\sum_{\mathbf{m},\kappa} \mathbf{m} \mathrm{Im}(\delta \tilde{H}^*_{\mathbf{m}}(\mathbf{J};\omega_\kappa) \delta \tilde{F}_{\mathbf{m}}(\mathbf{J};\omega_\kappa)) \right]. \end{aligned} \tag{6}$$

Substituting the Fourier decomposition Eq. (5) into Eq. (4), on the other hand, yields the formal solution for the Fourier component $\delta\tilde{F}_{\mathbf{m}}$:

$$\delta \tilde{F}_{\mathbf{m}}(\mathbf{J};\omega_\kappa) \equiv -(\omega_\kappa - \mathbf{m}\cdot\boldsymbol{\Omega})^{-1} \mathbf{m} \cdot \frac{\partial F_0}{\partial \mathbf{J}} \delta \tilde{H}_{\mathbf{m}}(\mathbf{J};\omega_\kappa), \tag{7}$$

where $\Omega_i(\mathbf{J}) \equiv \dot{\theta}_i = \partial H_0 / \partial J^i$ is the unperturbed orbital frequency associated with the fast orbital angle θ_i. Finally, substituting this solution into Eq. (6), the evolution equation (3) for F_0 becomes the quasilinear diffusion equation

$$\frac{\partial F_0}{\partial \tau} \equiv \frac{\partial}{\partial \mathbf{J}} \cdot \left(\mathsf{D}_{\mathrm{QL}} \cdot \frac{\partial F_0}{\partial \mathbf{J}} \right), \tag{8}$$

where the components of the quasilinear diffusion tensor are

$$D^{ij}_{\mathrm{QL}} \equiv \sum_{\mathbf{m},\kappa} m^i m^j [\pi \delta(\omega_\kappa - \mathbf{m}\cdot\boldsymbol{\Omega}) | \delta \tilde{H}_{\mathbf{m}}(\mathbf{J};\omega_\kappa)|^2], \tag{9}$$

and we have used $\mathrm{Im}(\omega_\kappa - \mathbf{m}\cdot\boldsymbol{\Omega})^{-1} \equiv -\pi\delta(\omega_\kappa - \mathbf{m}\cdot\boldsymbol{\Omega})$, which indicates that quasilinear diffusion is produced by resonant wave-particle interactions. The action-angle formulation of quasilinear transport theory therefore establishes the following paradigm: each wave-particle resonance introduces an explicit violation of the adiabatic invariance of an action variable which results in stochastic transport (e.g., bounce-resonant fluctuations destroy violate adiabatic bounce invariance and set up stochastic transport in bounce-action space).

We note that the singular character of the quasilinear diffusion tensor Eq. (9) disappears when nonlinear effects such as resonance broadening are taken into account.[11,13] In Eq. (9), for example, resonance broadening effects may be modeled by replacing the small denominator $(\omega_\kappa - \mathbf{m}\cdot\boldsymbol{\Omega})^{-1}$ with $(\omega_\kappa - \mathbf{m}\cdot\boldsymbol{\Omega} + i\nu)^{-1}$, where the autocorrelation time $\tau_{\mathrm{ac}} \equiv 1/\nu$ represents the time scale over which a resonant particle loses correlation with the waves.[13] Hence, using the prescription $\mathrm{Im}(\omega_\kappa - \mathbf{m}\cdot\boldsymbol{\Omega} + i\nu)^{-1} \equiv -\tau_{\mathrm{ac}}$ for resonant particles (i.e., $\omega_\kappa = \mathbf{m}\cdot\boldsymbol{\Omega}$), and the quasilinear diffusion tensor Eq. (9) is replaced with the *renormalized* quasilinear diffusion (RQL) tensor

$$D^{ij}_{\mathrm{RQL}} \equiv \left(\sum_{\mathbf{m},\kappa} m^i m^j | \delta \tilde{H}^{(\mathrm{res})}_{\mathbf{m}}|^2 \right) \tau_{\mathrm{ac}}, \tag{10}$$

where $\delta\tilde{H}^{(\mathrm{res})}_{\mathbf{m}} \equiv \delta\tilde{H}_{\mathbf{m}}(\mathbf{J};\omega_\kappa = \mathbf{m}\cdot\boldsymbol{\Omega})$. Such nonlinear effects are, however, beyond the scope of the present work and we postpone further discussion of nonlinear resonance-broadening effects to future work on applications of relativistic bounce-averaged quasilinear diffusion in magnetospheric physics.

B. Orbital time scales for relativistic electrons

In the present work, we focus our attention on axisymmetric magnetic geometry represented by the magnetic coordinates (ψ, φ, s), where the azimuthal angle φ is an ignorable coordinate whereas the magnetic flux ψ is a radial-like coordinate and s is the parallel coordinate along a magnetic-field line labeled by (ψ, φ). The background Vlasov distribution $F_0(J_g, \varepsilon, \psi)$ is a function of the relativistic gyro-action $J_g \equiv |\mathbf{p}_\perp|^2/(2M\omega_{\mathrm{g}})$ (an *adiabatic* invariant associated with the asymptotic elimination of the gyroangle ζ),[14] the particle energy ε [instead of the bounce action $J_b \equiv (1/2\pi) \oint p_\parallel \, ds$], and the radial-like drift action $J_d \equiv q\psi/c$ (an exact invariant associated with axisymmetry in φ).

The type of hydromagnetic fluctuations considered in this work are ultra-low-frequency (ULF) fluctuations in the frequency range 2–10 mHz. For a 1-MeV test electron at geosynchronous orbit,[15] the cyclotron frequency is in the range 1–3 kHz, the bounce frequency is in the range 1–3 Hz, and the drift frequency for equatorially-trapped electrons is in the range of a few mHz. Hence, a 1-MeV trapped electron can easily encounter magnetic-drift resonances with a ULF wave; hence, ULF waves can lead to radial transport as a result of the destruction of the third adiabatic invariant. Because of the time-scale ordering $\omega \ll \omega_c$ and $\omega_c \gg \omega_b \gg \omega_d$, the ULF fluctuations conserve the first adiabatic invariant, while the second and third invariants may be broken or not.

In low-frequency gyrokinetics,[14] the fluctuation time scale $(2\pi/\omega)$ is assumed much longer than the fast cyclotron time scale $(2\pi/\omega_c)$ and could be comparable to the intermediate bounce time scale $(2\pi/\omega_b)$ or the slow drift time scale $(2\pi/\omega_d)$. Because $\omega \ll \omega_g \equiv \gamma^{-1}\omega_c$, an asymptotic expression for the relativistic gyroaction J_g can be constructed which is conserved in the presence of low-frequency electromagnetic fluctuations. The condition $\omega \sim \omega_b$ or ω_d, on the other hand, implies that the construction of the bounce and drift invariants beyond their simplest expressions is unnecessary.

C. Relativistic drift-kinetic Vlasov equation

We now review the relativistic gyrokinetic Vlasov equation for low-frequency electromagnetic fluctuations in a nonuniform magnetized plasma with general magnetic geometry recently derived by Brizard and Chan.[14] Because electrons have negligible Larmor radii compared to the background scale lengths, we focus our attention on the drift-kinetic limit. For a particle species with mass M and charge q (particle species here is arbitrary), the relativistic drift-kinetic Vlasov equation is written in terms of the gyrocenter Vlasov distribution $F(\mathbf{X},p_\parallel,t;J_g)$ and the relativistic gyrocenter Hamiltonian $H(\mathbf{X},p_\parallel,t;J_g)$ as

$$\frac{\partial F}{\partial t}+\left(\frac{\mathbf{B}^*}{B_\parallel^*}\frac{\partial H}{\partial p_\parallel}+\frac{c\hat{\mathbf{b}}}{qB_\parallel^*}\times\nabla H\right)\cdot\nabla F-\frac{\mathbf{B}^*}{B^*}\cdot\nabla H\frac{\partial F}{\partial p_\parallel}=0, \tag{11}$$

where $\mathbf{B}^*\equiv\nabla\times[\mathbf{A}+(cp_\parallel/q)\hat{\mathbf{b}}]$, $\hat{\mathbf{b}}\equiv\mathbf{B}/B$, and $B_\parallel^*\equiv\hat{\mathbf{b}}\cdot\mathbf{B}^*$ are quantities derived from the unperturbed background magnetic field only. In the drift-kinetic limit, the linearized gyrocenter Hamiltonian H is expressed as

$$\begin{aligned}H&=(\gamma-1)Mc^2+\epsilon\left[q\left(\delta\phi-\frac{p_\parallel}{\gamma Mc}\delta A_\parallel\right)+J_g\,\omega_g\frac{\delta B_\parallel}{B}\right]\\&\equiv H_0+\epsilon\delta H,\end{aligned} \tag{12}$$

where $\omega_g\equiv|q|B/Mc$ is the rest-mass gyrofrequency and the relativistic factor is

$$\gamma(\mathbf{X},p_\parallel,J_g)\equiv\sqrt{1+\frac{2J_g\omega_g}{Mc^2}+\left(\frac{p_\parallel}{Mc}\right)^2}.$$

In Eqs. (11) and (12), the gyrocenter phase-space coordinates $(\mathbf{X},p_\parallel)$ are the gyrocenter position $\mathbf{X}$ and the relativistic gyrocenter parallel momentum $p_\parallel\equiv\gamma Mv_\parallel$, the relativistic gyroaction J_g is an invariant of gyrocenter Hamiltonian dynamics, and the background electric field is assumed to be negligible. In Eq. (12), the electromagnetic-field fluctuations are represented by the following scalar fields: $\delta\phi$ denotes the perturbed electrostatic scalar potential, while $\delta A_\parallel\equiv\hat{\mathbf{b}}\cdot\delta\mathbf{A}$ and $\delta B_\parallel\equiv\hat{\mathbf{b}}\cdot\delta\mathbf{B}$ denote the parallel components of the perturbed magnetic vector potential and the perturbed magnetic field, respectively.

Before proceeding to the derivation of the relativistic quasilinear drift-kinetic equations, we first perform a change of gyrocenter phase-space coordinates wherein we replace the relativistic gyrocenter parallel momentum $p_\parallel$ with the unperturbed relativistic gyrocenter particle energy $\varepsilon\equiv H_0$. We also introduce the unperturbed relativistic guiding-center velocity

$$\begin{aligned}\dot{\mathbf{X}}_0&\equiv\frac{\mathbf{B}^*}{B_\parallel^*}\frac{\partial H_0}{\partial p_\parallel}+\frac{c\hat{\mathbf{b}}}{qB_\parallel^*}\times\nabla H_0\\&=\frac{p_\parallel}{\gamma M}\hat{\mathbf{b}}+\frac{c\hat{\mathbf{b}}}{\gamma qB^*}\times\left(J_g\nabla\omega_g+\frac{p_\parallel^2}{M}\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}\right),\end{aligned} \tag{13}$$

where the first term in Eq. (13) represents parallel motion along a magnetic-field line, while the second and third terms represent grad-B and curvature magnetic drifts across field lines, respectively. After making the substitution $p_\parallel\to\varepsilon$ and using Eq. (13), the relativistic drift-kinetic Vlasov equation (11) becomes

$$\begin{aligned}\left(\frac{\partial}{\partial t}+\dot{\mathbf{X}}_0\cdot\nabla\right)F=\epsilon\bigg[&\frac{c\hat{\mathbf{b}}}{qB_\parallel^*}\cdot\nabla F\times\nabla\delta H\\&+\dot{\mathbf{X}}_0\cdot\left(\nabla\delta H\frac{\partial F}{\partial\varepsilon}-\nabla F\frac{\partial\delta H}{\partial\varepsilon}\right)\bigg].\end{aligned} \tag{14}$$

The physical interpretation of F in Eq. (14) is that it represents the phase-space number density of gyrocenters; its relationship to the particle phase-space density f is that $F(\mathbf{X},\varepsilon;J_g,\zeta;t)\equiv f(\mathbf{x},\mathbf{p};t)$ to all orders in ϵ.

Lastly, the Jacobian for the transformation from the particle phase-space coordinates $(\mathbf{x},\mathbf{p})$ to the reduced gyrocenter phase-space coordinates $(\mathbf{X},\varepsilon;J_g)$ is $(2\pi|q|/c)B_\parallel^*/|v_\parallel|$, where the factor 2π comes from the elimination of the gyroangle variable ζ. One can easily show from Eq. (13) that $\mathcal{D}\equiv B_\parallel^*/|v_\parallel|$ satisfies the identities

$$\nabla\cdot(\dot{\mathbf{X}}_0\,\mathcal{D})\equiv0\quad\text{and}\quad\nabla\times\left(\frac{c\mathcal{D}\hat{\mathbf{b}}}{qB_\parallel^*}\right)\equiv\frac{\partial}{\partial\varepsilon}(\dot{\mathbf{X}}_0\,\mathcal{D}). \tag{15}$$

For the axisymmetric magnetic geometry considered in this work, we find $B_\parallel^*\equiv B$ and we henceforth write $\mathcal{D}\equiv B/|v_\parallel|$ wherever it appears.

D. Organization

The remainder of the paper is organized as follows. In Sec. II, relativistic quasilinear drift-kinetic equations are derived in axisymmetric magnetic geometry in which the background fields are independent of the azimuthal angle φ. For this purpose, magnetic coordinates (ψ,φ,s) are introduced, where the magnetic flux ψ is an exact invariant of the unperturbed guiding-center motion and the parallel spatial coordinate s denotes a location along a magnetic field line labeled by (ψ,φ). The perturbed Vlasov distribution δF is then decomposed into the adiabatic part $(\delta H\partial_\varepsilon F_0)$ and the nonadiabatic part (denoted δG). Quasilinear drift-kinetic equations are subsequently written for the slow-time evolution of F_0 in terms of adiabatic and nonadiabatic contributions and the fast-time evolution of the nonadiabatic part of the perturbed Vlasov distribution δG. In Sec. III, the quasilinear evolution drift-kinetic equation for F_0 is bounce averaged and we show that only nonadiabatic contributions remain, i.e., under general assumptions, the adiabatic contributions to quasilinear diffusion vanish. In Sec. IV, we solve explicitly for the components of the relativistic bounce-averaged quasilinear diffusion tensor associated with stochastic transport in (ψ,ε) space. We summarize our work in Sec. V and discuss possible applications in magnetospheric physics. Lastly, Appendices A and B present additional details associated with general axisymmetric magnetic geometry (Appendix A) and the form of the perturbed electric and magnetic fields (Appendix B).

II. RELATIVISTIC QUASILINEAR DRIFT-KINETIC EQUATIONS IN AXISYMMETRIC MAGNETIC GEOMETRY

A. General relativistic quasilinear drift-kinetic equations

Relativistic quasilinear drift-kinetic equations can now be derived from Eq. (14) by first introducing the following decomposition on the gyrocenter Vlasov distribution:

$$F \equiv F_0 + \epsilon\,\delta F, \tag{16}$$

where the gyrocenter Vlasov distribution F_0 is defined as having a *slow* (transport) time scale dependence while the gyrocenter Vlasov distribution δF has a *fast* (wave) time scale dependence.

Next, we introduce the fast time scale and azimuthal-angle average with the property that $\overline{F_0} \equiv F_0$ and $\overline{\delta F} \equiv 0$; the notation $\overline{(\cdots)}$ is henceforth used to mean fast time scale and azimuthal-angle averaging only and appropriate notation for bounce averaging will be introduced later. Upon averaging Eq. (14), we obtain

$$\left(\epsilon^2 \frac{\partial}{\partial \tau} + \dot{\mathbf{X}}_0 \cdot \nabla\right) F_0 = \epsilon^2 \left[\dot{\mathbf{X}}_0 \cdot \overline{\left(\nabla\,\delta H \frac{\partial\,\delta F}{\partial \varepsilon} - \nabla\,\delta F \frac{\partial\,\delta H}{\partial \varepsilon}\right)} + \frac{c\hat{\mathbf{b}}}{qB_\parallel^*} \cdot \overline{(\nabla\,\delta F \times \nabla\,\delta H)} \right], \tag{17}$$

where $F_0 = F_0(\ldots, \epsilon^2 t \equiv \tau)$, i.e., the slow time scale dependence of F_0 enters at second order in ϵ. At order ϵ^0, Eq. (17) yields

$$\dot{\mathbf{X}}_0 \cdot \nabla F_0 \equiv 0, \tag{18}$$

i.e., F_0 is a constant along an unperturbed guiding-center orbit. At order ϵ^2, on the other hand, Eq. (17) yields

$$\frac{\partial F_0}{\partial \tau} = \left[\frac{c\hat{\mathbf{b}}}{qB_\parallel^*} \cdot \overline{(\nabla\,\delta F \times \nabla\,\delta H)} + \dot{\mathbf{X}}_0 \cdot \overline{\left(\nabla\,\delta H \frac{\partial\,\delta F}{\partial \varepsilon} - \nabla\,\delta F \frac{\partial\,\delta H}{\partial \varepsilon}\right)} \right]. \tag{19}$$

This quasilinear drift-kinetic equation states that the slow time scale relativistic gyrocenter Vlasov distribution F_0 evolves as a result of quadratic wave nonlinearities associated with the perturbed relativistic gyrocenter Vlasov distribution δF and the perturbed relativistic gyrocenter Hamiltonian δH. To derive a relativistic quasilinear drift-kinetic diffusion equation from Eq. (19), we must now establish a relation between δF and δH.

For this purpose, we consider the linearized relativistic drift-kinetic Vlasov equation. On the fast time scale (order ϵ), Eq. (14) yields

$$\left(\frac{\partial}{\partial t} + \dot{\mathbf{X}}_0 \cdot \nabla\right) \delta F = \left(\frac{c\hat{\mathbf{b}}}{qB_\parallel^*} \cdot \nabla F_0 \times \nabla\,\delta H + \dot{\mathbf{X}}_0 \cdot \nabla\,\delta H \frac{\partial F_0}{\partial \varepsilon} \right), \tag{20}$$

where Eq. (18) was used. We can rewrite Eq. (20) by decomposing the perturbed gyrocenter Vlasov distribution δF in terms of its *adiabatic* and *nonadiabatic* components,[16] respectively:

$$\delta F \equiv \delta H \frac{\partial F_0}{\partial \varepsilon} + \delta G, \tag{21}$$

where δG represents the nonadiabatic part of the perturbed gyrocenter Vlasov distribution. The linear relativistic drift-kinetic Vlasov equation for the nonadiabatic part δG is then obtained by substituting Eq. (21) into Eq. (20):

$$\left(\frac{\partial}{\partial t} + \dot{\mathbf{X}}_0 \cdot \nabla\right) \delta G = \left(\frac{c\hat{\mathbf{b}}}{qB_\parallel^*} \cdot \nabla F_0 \times \nabla - \frac{\partial F_0}{\partial \varepsilon} \frac{\partial}{\partial t} \right) \delta H. \tag{22}$$

This relativistic drift-kinetic equation generalizes the earlier nonrelativistic works of Antonsen and Lane,[17] Catto, Baldwin, and Tang,[18] and Chen.[19]

B. Axisymmetric magnetic geometry

In the present paper, we are interested in deriving quasilinear drift-kinetic equations (19) and (22) for an axisymmetric magnetic field geometry (however, note that deviations from axisymmetry can be incorporated as magnetic perturbations $\delta\mathbf{B}$ with zero frequency; see Appendices A and B for details). The axisymmetric magnetic field geometry is represented by the magnetic coordinates (ψ, φ, s), where

$$\mathbf{B} \equiv \nabla \psi \times \nabla \varphi \equiv B(\psi, s) \frac{\partial \mathbf{X}}{\partial s}. \tag{23}$$

Here, the magnetic flux ψ represents a radial-like coordinate, φ denotes the azimuthal angle, and s denotes the position of a point along a magnetic field line labeled by (ψ, φ). Note that the background magnetized plasma is assumed to be independent of the azimuthal angle φ. By definition, the parallel coordinate s is such that

$$\hat{\mathbf{b}} \equiv \nabla s + a\,\nabla \psi, \tag{24}$$

with $a(\psi, s) \equiv \hat{\mathbf{b}} \cdot \partial_\psi \mathbf{X}$, i.e., the magnetic coordinates (ψ, φ, s) are nonorthogonal coordinates. Moreover, from Eq. (24), we find $\nabla \times \hat{\mathbf{b}} = \hat{\mathbf{b}} \times (\partial_s a\,\nabla \psi)$, and thus $\hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}} = 0$ so that $B_\parallel^* \equiv B$. We note that in axisymmetric magnetic-dipole geometry, we have $\mathbf{B} = \nabla \chi(\psi, s)$, with $\partial_s \chi \equiv B$ and $\partial_\psi \chi \equiv a\,B$.

C. Perturbed magnetic and electric fields

The perturbed vector potential can be written in purely covariant form as

$$\delta\mathbf{A} \equiv \nabla\,\delta\alpha + \delta\psi\,\nabla\varphi - \delta\chi\,\nabla\psi, \tag{25}$$

where $\delta\alpha$, $\delta\psi$, and $\delta\chi$ are perturbation scalar potentials (see Appendix B for details). Note that the parallel component of the perturbed vector potential is $\delta A_\parallel \equiv \hat{\mathbf{b}} \cdot \delta\mathbf{A} = \partial_s\,\delta\alpha$. Hence, using Eq. (25), the perturbed magnetic field $\delta\mathbf{B} \equiv \nabla \times \delta\mathbf{A}$ can be expressed in explicit divergenceless form as

$$\delta\mathbf{B} \equiv \nabla\,\delta\psi \times \nabla\varphi + \nabla\psi \times \nabla\,\delta\chi. \tag{26}$$

Using the covariant representation Eq. (25) for the perturbed vector potential $\delta\mathbf{A}$, the perturbed electric field $\delta\mathbf{E} \equiv -\nabla\delta\phi - c^{-1}\,\partial_t\delta\mathbf{A}$, on the other hand, is written as

$$\delta\mathbf{E} = -\nabla\delta\Phi - \frac{1}{c}\left(\frac{\partial\delta\psi}{\partial t}\nabla\varphi - \frac{\partial\delta\chi}{\partial t}\nabla\psi\right), \tag{27}$$

where the scalar potentials $\delta\phi$ and $\delta\alpha$ are combined to define $\delta\Phi$ as follows:

$$\delta\Phi \equiv \delta\phi + \frac{1}{c}\frac{\partial\delta\alpha}{\partial t}. \tag{28}$$

Hence, the parallel component of the perturbed electric field is $\delta E_\parallel \equiv \hat{\mathbf{b}}\cdot\delta\mathbf{E} = -\,\partial_s\delta\Phi$.

D. Quasilinear equations in axisymmetric magnetic geometry

In axisymmetric magnetic geometry, the unperturbed guiding-center operator $\dot{\mathbf{X}}_0\cdot\nabla$ is thus written in terms of magnetic coordinates (ψ,φ,s) as

$$\dot{\mathbf{X}}_0\cdot\nabla = v_\parallel\frac{\partial}{\partial s} + \omega_d\,\frac{\partial}{\partial\varphi}, \tag{29}$$

where $v_\parallel \equiv p_\parallel/\gamma M$ is the parallel component of the relativistic guiding-center velocity and the magnetic-drift frequency is defined as

$$\omega_d(s;\varepsilon,\psi,J_g) \equiv \frac{c}{q\gamma}\left[J_g\overbrace{\left(\frac{\partial\omega_g}{\partial\psi} - a\frac{\partial\omega_g}{\partial s}\right)}^{\text{grad-B}} + \frac{p_\parallel^2}{M}\overbrace{\left(\frac{\partial a}{\partial s}\right)}^{\text{curvature}}\right]. \tag{30}$$

We note that since $\dot{\mathbf{X}}_0\cdot\nabla\psi = 0$, the drift-action $J_\mathrm{d} \equiv q\psi/c$ is an invariant for unperturbed particle motion in axisymmetric magnetic geometry. We also find from Eq. (18) that $F_0(\mathbf{I},\tau)$ depends on the three invariants $\mathbf{I} \equiv (\varepsilon,\psi,J_g)$ of unperturbed guiding-center motion in axisymmetric magnetic geometry.

Using the magnetic coordinates $(\psi,\varphi,s) \equiv \psi^i$ and the decomposition Eq. (21), the slow-time relativistic quasilinear drift-kinetic Vlasov equation (19) can be written in divergence form[20] as

$$\begin{aligned}\frac{\partial}{\partial\tau}(\mathcal{D}F_0) &= -\nabla\cdot\left[\mathcal{D}\dot{\mathbf{X}}_0\overline{\left(\delta F\frac{\partial\delta H}{\partial\varepsilon}\right)} + \frac{c\mathcal{D}\hat{\mathbf{b}}}{qB}\times\overline{(\delta F\nabla\delta H)}\right] \\ &\quad + \frac{\partial}{\partial\varepsilon}\left[\mathcal{D}\dot{\mathbf{X}}_0\cdot\overline{(\delta F\nabla\delta H)}\right] \\ &\equiv -B\frac{\partial}{\partial\psi^i}\left[\frac{\mathcal{D}}{B}\left(R^i + D_A^{i\varepsilon}\frac{\partial F_0}{\partial\varepsilon}\right)\right] \\ &\quad + \frac{\partial}{\partial\varepsilon}\left[\mathcal{D}\left(R^\varepsilon + D_A^{\varepsilon\varepsilon}\frac{\partial F_0}{\partial\varepsilon}\right)\right],\end{aligned} \tag{31}$$

where the adiabatic (A) coefficients $(D_A^{i\varepsilon}, D_A^{\varepsilon\varepsilon})$ are defined

$$D_A^{i\varepsilon} \equiv \frac{1}{2}\left(\frac{\partial\overline{\delta H^2}}{\partial\varepsilon}\dot{\mathbf{X}}_0 + \frac{c\hat{\mathbf{b}}}{qB}\times\nabla\overline{\delta H^2}\right)\cdot\nabla\psi^i$$

and (32)

$$D_A^{\varepsilon\varepsilon} \equiv \tfrac{1}{2}\dot{\mathbf{X}}_0\cdot\nabla\overline{\delta H^2},$$

while the nonadiabatic coefficients (R^i,R^ε) are defined

$$R^i \equiv \nabla\psi^i\cdot\overline{\left[\delta G\left(\dot{\mathbf{X}}_0\frac{\partial\delta H}{\partial\varepsilon} + \frac{c\hat{\mathbf{b}}}{qB}\times\nabla\delta H\right)\right]}$$

and (33)

$$R^\varepsilon \equiv \dot{\mathbf{X}}_0\cdot\overline{(\delta G\nabla\delta H)}.$$

The linear relativistic drift-kinetic Vlasov equation (22) for δG can be written in terms of the magnetic coordinates (ψ,φ,s) as

$$\left(\frac{\partial}{\partial t} + v_\parallel\frac{\partial}{\partial s} + \omega_d\frac{\partial}{\partial\varphi}\right)\delta G = \left(\frac{c}{q}\frac{\partial F_0}{\partial\psi}\frac{\partial}{\partial\varphi} - \frac{\partial F_0}{\partial\varepsilon}\frac{\partial}{\partial t}\right)\delta H, \tag{34}$$

where the total time derivative appearing on the left side contains bounce and magnetic-drift contributions. Although the adiabatic coefficients Eq. (32) are shown in the next section to vanish under bounce averaging, we keep them here for completeness.

III. RELATIVISTIC BOUNCE-AVERAGED QUASILINEAR DRIFT-KINETIC DIFFUSION EQUATION

Since the time-independent background medium is axisymmetric in φ, we decompose the perturbation fields in Fourier components in time t and azimuthal angle φ:

$$\begin{pmatrix}\delta G\\ \delta H\end{pmatrix} \equiv \sum_\kappa\sum_{m=-\infty}^{\infty}\begin{pmatrix}\delta\tilde{G}_m(s;\mathbf{I},\omega_\kappa)\\ \delta\tilde{H}_m(s;\mathbf{I},\omega_\kappa)\end{pmatrix}e^{(im\varphi-\omega_\kappa t)}. \tag{35}$$

In what follows, we shall use the notation $\tilde{f}$ to indicate explicit s-dependence in a function f and the notation $\hat{f}$ to indicate s-independence (i.e., $\partial_s\hat{f} \equiv 0$). For two arbitrary functions $f(\mathbf{X},t)$ and $g(\mathbf{X},t)$, the fast time scale average $\overline{(fg)}$ is now explicitly written as

$$\overline{(fg)} \equiv \sum_{m,\kappa}\tilde{f}_m^*(\psi,s;\omega_\kappa)\tilde{g}_m(\psi,s;\omega_\kappa). \tag{36}$$

Thus we find from Eqs. (32) and (35) that $D_A^{\psi\varepsilon} \equiv 0$ and $D_A^{\varepsilon\varepsilon} \equiv \frac{1}{2}v_\parallel\,\partial_s\overline{\delta H^2}$.

Next, since F_0 is independent of the parallel coordinate s, we introduce a bounce-average operation defined as

$$\langle f\rangle(\mathbf{I}) \equiv \frac{1}{\tau_b}\sum_{\sigma=\pm1}\int_{s_L}^{s_U}\frac{ds}{|v_\parallel|}f(s,\sigma;\mathbf{I}), \tag{37}$$

where the sum is over the two possible signs of $v_\parallel$ (note that $\mathcal{D}/B \equiv 1/|v_\parallel|$), and the bounce period is defined as

$$\tau_\mathrm{b}(\mathbf{I}) \equiv \sum_{\sigma=\pm1}\int_{s_L}^{s_U}\frac{\mathcal{D}}{B}ds = 2\int_{s_L}^{s_U}\frac{ds}{|v_\parallel|} \equiv 2\pi\frac{\partial J_b}{\partial\varepsilon},$$

with $s_L(\mathbf{I})$ and $s_U(\mathbf{I})$ denoting the lower (L) and upper (U) turning points. We note that since F_0 is independent of the parallel spatial coordinate s, we have $\langle F_0\rangle \equiv F_0$.

We now perform the bounce-average operation on Eq. (31) to obtain the relativistic quasilinear bounce-averaged drift-kinetic diffusion equation:[21]

$$\frac{\partial F_0}{\partial \tau} \equiv \frac{1}{\tau_b}\frac{\partial}{\partial \psi}\left[\tau_b\left\langle \frac{c\hat{\mathbf{b}}}{qB}\times\nabla\psi\cdot\overline{(\delta G\nabla\delta H)}\right\rangle\right] + \frac{1}{\tau_b}\frac{\partial}{\partial \varepsilon}[\tau_b\langle \dot{\mathbf{X}}_0\cdot\overline{(\delta G\,\nabla\,\delta H)}\rangle], \tag{38}$$

where the s-term in Eq. (31) is averaged out while the φ-term vanishes as a result of axisymmetry. Here, we used $\dot{\mathbf{X}}_0\cdot\nabla\psi = 0$ in the first term in Eq. (38) while the adiabatic coefficient $D_A^{\varepsilon\varepsilon}$ vanishes upon bounce-averaging since

$$\langle D_A^{\varepsilon\varepsilon}\rangle = \frac{1}{2}\langle \dot{\mathbf{X}}_0\cdot\nabla\overline{\delta H^2}\rangle = \frac{1}{2}\left\langle v_\parallel \frac{\partial\overline{\delta H^2}}{\partial s}\right\rangle \equiv 0.$$

Hence, only nonadiabatic effects contribute to quasilinear diffusion.

Using the Fourier decomposition Eq. (35) in the linear relativistic drift-kinetic Vlasov equation (34) for the nonadiabatic part of the perturbed Vlasov distribution, we obtain

$$\mathcal{L}\delta\tilde{G}_m(\sigma,s) \equiv i\,\mathcal{F}\delta\tilde{H}_m(\sigma,s), \tag{39}$$

where $\mathcal{F}(\mathbf{I},\omega_\kappa) \equiv \omega_\kappa\,\partial\varepsilon F_0 + (mc/q)\,\partial_\psi F_0$ and we henceforth omit displaying the dependence of the perturbation fields on $\mathbf{I}$ and ω_κ. In Eq. (39), the differential operator $\mathcal{L}$ is defined as

$$\mathcal{L}\delta\tilde{G}_m(\sigma,s) \equiv \left[\sigma|v_\parallel|\frac{\partial}{\partial s} - i\,(\omega_\kappa - m\,\omega_{\mathrm{d}})\right]\delta\tilde{G}_m(\sigma,s), \tag{40}$$

and the perturbed Hamiltonian defined in Eq. (12) becomes

$$\delta\tilde{H}_m(\sigma,s) = q\left(\delta\tilde{\phi}_m(s) - \sigma\frac{|v_\parallel|}{c}\,\delta\tilde{A}_{\parallel m}(s)\right) + J_g\omega_g\frac{\delta\tilde{B}_{\parallel m}(s)}{B}. \tag{41}$$

The operator $\mathcal{L}$ defined in Eq. (40) has the following property associated with bounce-averaging operation defined in Eq. (37):

$$\langle f^*(\mathcal{L}g)\rangle = -\langle(\mathcal{L}f)^*g\rangle \tag{42}$$

for two arbitrary functions f and g. Using Eq. (35) in Eq. (38), we therefore find

$$\langle \dot{\mathbf{X}}_0\cdot\overline{(\delta G\nabla\delta H)}\rangle \equiv \sum_{m,\kappa}\omega_\kappa\,\mathrm{Im}\langle\delta\tilde{G}_m\delta\tilde{H}_m^*\rangle, \tag{43}$$

and

$$\left\langle \frac{\hat{\mathbf{b}}}{B}\times\nabla\psi\cdot\overline{(\delta G\nabla\delta H)}\right\rangle \equiv \sum_{m,\kappa} m\,\mathrm{Im}\langle\delta\tilde{G}_m\,\delta\tilde{H}_m^*\rangle. \tag{44}$$

By introducing the *quasilinear* perturbation potential

$$\hat{\Gamma}_m(\mathbf{I};\omega_\kappa) \equiv \mathcal{F}^{-1}\,\mathrm{Im}\langle\delta\tilde{G}_m\,\delta\tilde{H}_m^*\rangle, \tag{45}$$

the relativistic quasilinear bounce-averaged drift-kinetic diffusion equation (38) now becomes

$$\frac{\partial F_0}{\partial\tau} \equiv \frac{1}{\tau_{\mathrm{b}}}\frac{\partial}{\partial\varepsilon}\left[\tau_b\left(D_{\mathrm{QL}}^{\varepsilon\varepsilon}\frac{\partial F_0}{\partial\varepsilon} + D_{\mathrm{QL}}^{\varepsilon\psi}\frac{\partial F_0}{\partial\psi}\right)\right] + \frac{1}{\tau_b}\frac{\partial}{\partial\psi}\left[\tau_{\mathrm{b}}\left(D_{\mathrm{QL}}^{\psi\varepsilon}\frac{\partial F_0}{\partial\varepsilon} + D_{\mathrm{QL}}^{\psi\psi}\frac{\partial F_0}{\partial\psi}\right)\right], \tag{46}$$

where the relativistic quasilinear bounce-averaged diffusion coefficients are defined as

$$D_{\mathrm{QL}}^{\varepsilon\varepsilon}(\mathbf{I}) \equiv \sum_{m,\kappa}\omega_\kappa^2\hat{\Gamma}_m(\mathbf{I};\omega_\kappa),$$

$$D_{\mathrm{QL}}^{\psi\varepsilon}(\mathbf{I}) \equiv \sum_{m,\kappa}(mc/q)\,\omega_\kappa\hat{\Gamma}_m(\mathbf{I};\omega_\kappa) \equiv D_{\mathrm{QL}}^{\varepsilon\psi}(\mathbf{I}), \tag{47}$$

$$D_{\mathrm{QL}}^{\psi\psi}(\mathbf{I}) \equiv \sum_{m,\kappa}(mc/q)^2\hat{\Gamma}_m(\mathbf{I};\omega_\kappa).$$

This quasilinear diffusion tensor describes stochastic transport in (ψ,ε)-space induced by low-frequency electromagnetic fluctuations which conserve the first (adiabatic) invariant J_g. We now proceed to obtain an explicit expression for $\hat{\Gamma}_m$ in terms of the perturbation Hamiltonian $\delta\tilde{H}_m$.

IV. SOLUTION OF THE NONADIABATIC DRIFT-KINETIC EQUATION FOR TRAPPED PARTICLES

To obtain an explicit expression for the quasilinear perturbation potential $\hat{\Gamma}_m$, we need to solve the linearized relativistic drift-kinetic equation (39) for the nonadiabatic part $\delta\tilde{G}_m(\sigma,s)$ of the perturbed gyrocenter Vlasov distribution in terms of the perturbed relativistic gyrocenter Hamiltonian $\delta\tilde{H}_m(\sigma,s)$.

The fact that the perturbed relativistic gyrocenter Hamiltonian Eq. (41) depends on σ introduces some difficulties in solving Eq. (39). The solution of Eq. (39) can, however, be simplified as follows. First, using $\delta A_\parallel \equiv \hat{\mathbf{b}}\cdot\delta\mathbf{A} = \partial\delta\alpha/\partial s$ from Eq. (25), we can write $\delta\tilde{H}_m$ as

$$\delta\tilde{H}_m(\sigma,s) \equiv \delta\tilde{K}_m(s) - \frac{q}{c}\mathcal{L}\delta\tilde{\alpha}_m, \tag{48}$$

with

$$\begin{aligned}\delta\tilde{K}_m(s) &\equiv q\,\delta\tilde{\phi}_m + J_g\omega_g(\delta\tilde{B}_{\parallel m}/B) - i\frac{q}{c}(\omega_\kappa - m\,\omega_d)\,\delta\tilde{\alpha}_m\\ &\equiv q\,\delta\tilde{\Phi}_m + J_g\omega_g(\delta\tilde{B}_{\parallel m}/B) + \frac{imq}{c}\omega_d\,\delta\tilde{\alpha}_m,\end{aligned} \tag{49}$$

where $\delta\tilde{\Phi}_m \equiv \delta\tilde{\phi}_m - i(\omega_\kappa/c)\,\delta\tilde{\alpha}_m$ [see Eq. (28)]; the reader is referred to Appendix B for a representation of $\delta\hat{K}_m$ in terms of components of the perturbed electric and magnetic fields. When the decomposition Eq. (48) is inserted into Eq. (39), we obtain the new equation

$$\mathcal{L}\delta\tilde{G}'_m(\sigma,s) \equiv i\mathcal{F}\delta\tilde{K}_m(s), \tag{50}$$

where the new nonadiabatic part $\delta\tilde{G}'_m(\sigma,s)$ is defined as

$$\delta\tilde{G}_m \equiv \delta\tilde{G}'_m - i\,\frac{q}{c}\mathcal{F}\,\delta\tilde{\alpha}_m. \tag{51}$$

Using properties Eq. (42) of $\mathcal{L}$ and the decompositions Eqs. (48) and (51), we can show that the quasilinear perturbation potential Eq. (45) can also be written as

$$\hat{\Gamma}_m = \mathcal{F}^{-1}\,\mathrm{Im}\langle\,\delta\tilde{G}'_m\,\delta\tilde{K}^*_m\rangle, \tag{52}$$

in terms of $\delta\tilde{G}'_m$ and $\delta\tilde{K}_m$.

A. General solution for $\delta\tilde{G}'_m(\sigma,s)$

To proceed with the solution of the linearized drift-kinetic equation (50) we first write the operator $\mathcal{L}$ [defined in Eq. (39)] in terms of the integrating factor θ:

$$\mathcal{L}\,\delta\tilde{G}'_m(\sigma,s) \equiv e^{i\sigma\theta(s)}v_\parallel\frac{\partial}{\partial s}[e^{-i\sigma\theta(s)}\delta\tilde{G}'_m(\sigma,s)],$$

where $\theta(s)$ is defined as

$$\theta(s_L,s) \equiv \int_{s_L}^{s}\frac{ds'}{|v_\parallel|}(\omega_\kappa - m\,\omega_d). \tag{53}$$

Hence, Eq. (50) can be written as

$$e^{i\sigma\theta(s)}v_\parallel\frac{\partial}{\partial s}[e^{-i\sigma\theta(s)}\delta\tilde{G}'_m(\sigma,s)] = i\mathcal{F}\delta\tilde{K}_m(s), \tag{54}$$

which implies that its general solution $\delta\tilde{G}'_m(\sigma,s)$ (for trapped particles with turning points at $s=s_L$ and $s=s_U$) can be written as

$$\delta\tilde{G}'_m(\sigma,s) \equiv \mathcal{F}e^{i\sigma\theta(s)}[\,\delta\hat{g}_m(\sigma) + \delta\tilde{g}_m(\sigma,s)]. \tag{55}$$

The homogeneous term $\delta\hat{g}_m(\sigma)$ is determined from two matching conditions for $\delta\tilde{G}'_m(\sigma,s)$ for $\sigma=\pm 1$ at $s=s_L$ and $s=s_U$ (see below).

1. Inhomogeneous term

We first solve for the inhomogeneous term $\delta\tilde{g}_m(\sigma,s)$. After substituting the inhomogeneous term $\mathcal{F}\delta\tilde{g}_m\exp(i\sigma\theta)$ into Eq. (54), we find $\sigma|v_\parallel|\,\partial_s\delta\tilde{g}_m \equiv i\,\delta\tilde{K}_m\,e^{-i\sigma\theta}$. This equation is solved, subject to the boundary condition $\delta\tilde{g}_m(\sigma,s=s_L)\equiv 0$, as

$$\begin{aligned}\delta\tilde{g}_m(\sigma,s) &\equiv i\sigma\int_{s_L}^{s}\frac{ds'}{|v_\parallel|}e^{-i\sigma\theta(s_L,s')}\delta\tilde{K}_m(s') \\ &\equiv i\sigma\,\delta\tilde{k}_m(\sigma,s).\end{aligned} \tag{56}$$

By definition, the s-dependent function $\delta\tilde{k}_m(\sigma,s)$ satisfies the following relations: $\delta\tilde{k}_m(\sigma,s=s_L)\equiv 0$ and

$$\begin{aligned}\delta\tilde{k}_m(\sigma,s=s_U) &= \frac{\tau_b}{2}\langle\,\delta\tilde{K}_m e^{-i\sigma\theta}\rangle \\ &\equiv \frac{\tau_b}{2}(\langle\,\delta\tilde{K}_m\cos\theta\rangle - i\sigma\,\langle\,\delta\tilde{K}_m\sin\theta\rangle).\end{aligned} \tag{57}$$

2. Homogeneous term

We now solve for the homogeneous term $\delta\hat{g}_m(\sigma)$. The matching condition $\delta\tilde{G}'_m(+1,s_L) = \delta\tilde{G}'_m(-1,s_L)$ at the turning point $s=s_L$ yields $\delta\hat{g}_m(+1) = \delta\hat{g}_m(-1) \equiv \delta\hat{g}_m$, while the matching condition $\delta\tilde{G}'_m(+1,s_U) = \delta\tilde{G}'_m(-1,s_U)$ at the turning point $s=s_U$ yields

$$\begin{aligned}& e^{i\Theta}\left[\,\delta\hat{g}_m + i\,\frac{\tau_b}{2}\langle e^{-i\theta}\delta\tilde{K}_m\rangle\right] \\ &\quad = e^{-i\Theta}\left[\,\delta\hat{g}_m - i\,\frac{\tau_b}{2}\langle e^{i\theta}\delta\tilde{K}_m\rangle\right],\end{aligned} \tag{58}$$

where Eq. (57) was used and the phase factor $\Theta \equiv \theta(s_L,s_U)$ is defined from Eq. (53) as

$$\Theta = \frac{\tau_b}{2}(\omega_\kappa - m\,\langle\omega_d\rangle) \equiv \frac{\pi}{\omega_b}(\omega_\kappa - m\,\langle\omega_d\rangle). \tag{59}$$

The matching condition Eq. (58) can now be used to solve for the s-independent function $\delta\hat{g}_m$:

$$\delta\hat{g}_m \equiv -\,\frac{\tau_b}{2}(\cot\Theta\langle\,\delta\tilde{K}_m\cos\theta\rangle + \langle\,\delta\tilde{K}_m\sin\theta\rangle). \tag{60}$$

By combining the inhomogeneous solution Eq. (57) and the homogeneous solution Eq. (60), the general solution of Eq. (50) is therefore written as

$$\begin{aligned}\delta\tilde{G}'_m(\sigma,s) = \mathcal{F}e^{i\sigma\theta(s)}\Big[\, & i\sigma\,\delta\tilde{k}_m(\sigma,s) \\ & - \frac{\tau_b}{2}(\cot\Theta\langle\,\delta\tilde{K}_m\cos\theta\rangle + \langle\,\delta\tilde{K}_m\sin\theta\rangle)\Big].\end{aligned} \tag{61}$$

B. Quasilinear perturbation potential

The quasilinear perturbation potential Eq. (52) can now be evaluated explicitly. First, using Eq. (61) and the bounce-average definition Eq. (37), we obtain

$$\begin{aligned}\hat{\Gamma}_m &= \mathrm{Im}\langle\,\delta\tilde{K}^*_m\,e^{i\sigma\theta}(\,\delta\hat{g}_m + i\sigma\,\delta\tilde{k}_m)\rangle \\ &= \mathrm{Im}(\,\delta\hat{g}_m\langle\,\delta\tilde{K}^*_m\cos\theta\rangle) + \mathrm{Re}\left\langle\,\sigma\,|v_\parallel|\frac{\partial\,\delta\tilde{k}^*_m}{\partial s}\,\delta\tilde{k}_m\right\rangle,\end{aligned} \tag{62}$$

where, in the first term, we have used the fact that

$$\langle\,\sigma\,\delta\tilde{K}^*_m\sin\theta\rangle = \sum_{\sigma=\pm 1}\sigma\overbrace{\left(\frac{1}{\tau_b}\int_{s_L}^{s_U}\frac{ds}{|v_\parallel|}\,\delta\tilde{K}^*_m\sin\theta\right)}^{\text{independent of }\sigma} \equiv 0$$

and, in the second term, we have used Eq. (56) to obtain $\delta\tilde{K}^*_m\,e^{i\sigma\theta} \equiv |v_\parallel|\,\partial\,\delta\tilde{k}^*_m/\partial s$. Next, using Eq. (60), the first term in Eq. (62) becomes

$$\mathrm{Im}(\delta\hat{g}_m\langle\delta\tilde{K}_m^*\cos\theta\rangle)=-\frac{\tau_b}{2}|\langle\delta\tilde{K}_m\cos\theta\rangle|^2\mathrm{Im}(\cot\Theta)+\frac{\tau_b}{2}\mathrm{Im}(\langle\delta\tilde{K}_m\cos\theta\rangle\langle\delta\tilde{K}_m^*\sin\theta\rangle). \tag{63}$$

The second term in Eq. (62), on the other hand, becomes

$$\begin{aligned}\mathrm{Re}\left\langle\sigma|v_\parallel|\frac{\partial\delta\tilde{k}_m^*}{\partial s}\delta\tilde{k}_m\right\rangle&=\frac{1}{2}\left\langle\sigma|v_\parallel|\frac{\partial|\delta\tilde{k}_m|^2}{\partial s}\right\rangle\\&=\frac{1}{2\tau_b}\sum_\sigma\sigma\int_{s_L}^{s_U}\frac{ds}{|v_\parallel|}|v_\parallel|\frac{\partial|\delta\tilde{k}_m|^2}{\partial s}\\&=\frac{1}{2\tau_b}[|\delta\tilde{k}_m(\sigma=+1,s_U)|^2\\&\quad-|\delta\tilde{k}_m(\sigma=-1,s_U)|^2],\end{aligned} \tag{64}$$

where we have used the fact that $\delta\tilde{k}_m(\sigma,s_L)\equiv 0$ (by definition). Using Eq. (57) and the identity (for arbitrary complex functions a and b) $|a\pm i\sigma b|^2=|a|^2+|b|^2\pm 2\sigma\,\mathrm{Im}(ab^*)$, the inhomogeneous term Eq. (64) finally becomes

$$\mathrm{Re}\left\langle\sigma|v_\parallel|\frac{\partial\delta\tilde{k}_m^*}{\partial s}\delta\tilde{k}_m\right\rangle=-\frac{\tau_b}{2}\mathrm{Im}(\langle\delta\tilde{K}_m\cos\theta\rangle\langle\delta\tilde{K}_m^*\sin\theta\rangle), \tag{65}$$

which then cancels the second term in Eq. (63). By combining the homogeneous and inhomogeneous terms, the quasilinear perturbation potential Eq. (62) therefore becomes

$$\hat{\Gamma}_m\equiv\frac{\tau_b}{2}|\langle\delta\tilde{K}_m\cos\theta\rangle|^2\mathrm{Im}(-\cot\Theta). \tag{66}$$

We now show that the quasilinear diffusion of magnetically-trapped particles arises from *resonant* magnetically-trapped particles only. Using the identity $\cot\Theta=\Sigma_{n=-\infty}^{\infty}(\Theta-n\pi)^{-1}$ (for $\Theta\neq 0,\pm\pi,\ldots$), we obtain, after substituting Eq. (59), the expression

$$\cot\Theta\equiv\frac{1}{\pi}\sum_{n=-\infty}^{\infty}\frac{\omega_b}{(\omega_\kappa-m\langle\omega_d\rangle-n\,\omega_b)}, \tag{67}$$

where $\omega_b\equiv 2\pi/\tau_b$ is the bounce frequency. We note that the denominator in Eq. (67) is a function of the invariants $\mathbf{I}\equiv(\varepsilon,\psi,J_g)$ with (ω_κ,m,n) appearing as parameters. Using the Plemelj formula[22]

$$\lim_{\eta\to 0^+}(x-a+i\eta)^{-1}\equiv\mathcal{P}[(x-a)^{-1}]-i\pi\,\delta(x-a),$$

we readily find $\mathrm{Im}(x-a)^{-1}\equiv-\pi\,\delta(x-a)$. Hence, by applying the Plemelj formula in Eq. (67), we obtain the following expression for $\mathrm{Im}(-\cot\Theta)$:

$$\mathrm{Im}(-\cot\Theta)\equiv\sum_{n=-\infty}^{\infty}\omega_b\delta(\omega_\kappa-m\langle\omega_d\rangle-n\,\omega_b), \tag{68}$$

and the final expression for $\hat{\Gamma}_m$ is therefore

$$\hat{\Gamma}_m(\mathbf{I};\omega_\kappa)\equiv\sum_{n=-\infty}^{\infty}\pi\,\delta(\omega_\kappa-m\langle\omega_d\rangle-n\,\omega_b)\times|\langle\delta\tilde{K}_m(s;\mathbf{I},\omega_\kappa)\cos\theta(s;\mathbf{I},\omega_\kappa)\rangle|^2. \tag{69}$$

This expression contains contributions from magnetic-drift resonances ($\omega_\kappa\sim m\langle\omega_d\rangle$) and bounce resonances ($\omega_\kappa\sim n\omega_b$).

C. Relativistic bounce-averaged quasilinear diffusion tensor

Substituting Eq. (69) into Eq. (47), the components of the relativistic quasilinear bounce-averaged quasilinear diffusion tensor in (ψ,ε)-space are now expressed as

$$\begin{aligned}D_{\mathrm{QL}}^{\varepsilon\varepsilon}&\equiv\sum_{m,n,\kappa}\omega_\kappa^2[\pi\delta(\omega_\kappa-m\langle\omega_d\rangle-n\,\omega_b)\times|\langle\delta\tilde{K}_m\cos\theta\rangle|^2],\\D_{\mathrm{QL}}^{\psi\varepsilon}&\equiv\sum_{m,n,\kappa}(mc/q)\,\omega_\kappa[\pi\delta(\omega_\kappa-m\langle\omega_d\rangle-n\,\omega_b)\times|\langle\delta\tilde{K}_m\cos\theta\rangle|^2]\equiv D_{\mathrm{QL}}^{\varepsilon\psi},\\D_{\mathrm{QL}}^{\psi\psi}&\equiv\sum_{m,n,\kappa}(mc/q)^2[\pi\delta(\omega_\kappa-m\langle\omega_d\rangle-n\,\omega_b)\times|\langle\delta\tilde{K}_m\cos\theta\rangle|^2].\end{aligned} \tag{70}$$

As expected, these expressions are very similar in form to those obtained from general quasilinear theory [see Eq. (9)]. In the equatorial plane (where $\theta=\pi$), the term $|\delta\tilde{K}_m|^2$ is directly proportional to the energy spectral density of the waves. For particles which mirror off the equator, the diffusion tensor elements are calculated by bounce-averaging an s-dependent energy spectral density weighted by the factor $\cos\theta$, as shown in Eq. (70).

V. SUMMARY AND DISCUSSION

In this paper a relativistic quasilinear bounce-averaged drift-kinetic diffusion equation [Eq. (46)] has been derived and explicit expressions for the diffusion tensor elements have been obtained [Eq. (70)].

In the Introduction, beginning with the relativistic Vlasov equation in Poisson-bracket form, the standard two time scale decomposition of quasilinear theory is made, action-angle variables are introduced, and the solution of the linear Vlasov equation in action-angle form is substituted into the F_0 evolution equation and averaged over all fast variables to obtain the general action form of the relativistic quasilinear diffusion equation (8) and the corresponding diffusion tensor elements [Eq. (9)].

The derivation of Eqs. (46) and (70) follows steps analogous to the general action-angle derivation, but with important differences because of the use of different guiding-center phase-space coordinates and magnetic coordinates. The starting point is the relativistic drift-kinetic equation (11) obtained from the corresponding relativistic gyrokinetic equation derived recently by Brizard and Chan.[14] After changing from the $(\mathbf{X},p_\parallel;J_g)$ phase-space coordinates to $(\mathbf{X},\varepsilon;J_g)$ coordinates the two time scale quasilinear decomposition is introduced and an F_0-evolution equation and linear drift-kinetic equation are obtained [Eqs. (19) and (22)]. Next, magnetic coordinates (ψ,φ,s) are introduced via Eq. (23) and a bounce-averaged equation (38) for F_0 and a linearized relativistic drift-kinetic equation (39) are obtained. After introducing a quasilinear perturbation potential $\hat{\Gamma}_m$ [Eq. (45)], the final form of the relativistic quasilinear bounce-averaged drift-kinetic equation (46) is obtained. Finally, in Sec. IV the linearized relativistic drift-kinetic equation (39) is solved and the results are used to yield explicit expressions for the quasilinear diffusion tensor elements in Eq. (70).

By Eq. (18) and the discussion following Eq. (30), F_0 is independent of gyrophase, distance along the field line s, and azimuthal angle φ, so to lowest order F_0 can be regarded as being averaged over these three variables. Thus Eq. (46) describes slow, diffusive changes in a gyro-, bounce-, and drift-averaged phase-space density due to breaking of the second and third adiabatic invariants by drift-bounce wave-particle resonances associated with ultra-low-frequency (ULF) electromagnetic fluctuations (which, however, conserve the first adiabatic invariant). We note that ULF fluctuations associated with radial and energy quasilinear diffusion are not necessarily associated with strong pitch-angle scattering (which moves the particle mirror points along field lines[5,23]).

Lastly, we have remarked in the Introduction that the singular character of the quasilinear diffusion tensor disappears when nonlinear effects are taken into account.[11,13] Analogous expressions exist for the relativistic bounce-averaged quasilinear diffusion tensor Eq. (70), and these expressions will be used in future work to calculate renormalized diffusion coefficients for the drift-resonant transport of magnetospheric relativistic electrons in the presence of low-frequency hydromagnetic waves.

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APPENDIX A: AXISYMMETRIC MAGNETIC GEOMETRY

Magnetic field geometry is defined in terms of magnetic coordinates $\psi^i \equiv (\psi,\varphi,s)$, where $\mathbf{B}\equiv\nabla\psi\times\nabla\varphi$ and $\hat{\mathbf{b}}\equiv\mathbf{B}/B=\partial_s\mathbf{X}$. In axisymmetric magnetic-field geometry, where the azimuthal angle φ is an ignorable coordinate, the magnetic field is represented in contravariant form as $\mathbf{B}\equiv B(\psi,s)\partial s\mathbf{X}$, where s denotes the spatial coordinate along a magnetic-field line labeled by (ψ,φ). By definition, $\nabla s\cdot\nabla\psi\times\nabla\varphi\equiv B$ and the Jacobian for the transformation $\mathbf{X}\rightarrow(\psi,\varphi,s)$ is B^{-1}, i.e., $d^3X=B^{-1}\,d\psi\,d\varphi\,ds$.

The magnetic geometry is further defined by the components of the two-contravariant metric tensor $g^{ij}\equiv\nabla\psi^i\cdot\nabla\psi^j$ and the two-covariant metric tensor $g_{ij}\equiv\partial_i\mathbf{X}\cdot\partial_j\mathbf{X}$. These tensors are used in the following relations:

$$\partial_i\mathbf{X}\equiv g_{ij}\nabla\psi^j \quad\text{and}\quad \nabla\psi^i\equiv g^{ij}\,\partial_j\mathbf{X}, \tag{A1}$$

where summation over repeated indices is implied. We also find $\det(\mathbf{g}_{ij})\equiv B^{-2}$ and $\det(\mathbf{g}^{ij})\equiv B^2$, and we note that $g_i^j\equiv\partial_i\mathbf{X}\cdot\nabla\psi^j\equiv\delta_i^j$. For axisymmetric magnetic-field geometry, the components of the two-covariant metric tensor are given explicitly as

$$\begin{aligned}\partial_s\mathbf{X}&=\nabla s+a\,\nabla\psi\equiv\hat{\mathbf{b}},\\ \partial_\psi\mathbf{X}&=(\nabla\psi/|\nabla\psi|^2)+a\,\hat{\mathbf{b}},\\ \partial_\varphi\mathbf{X}&=R^2\nabla\varphi,\end{aligned} \tag{A2}$$

where $R\equiv|\nabla\varphi|^{-1}$ and $a(\psi,s)\equiv\hat{\mathbf{b}}\cdot\partial_\psi\mathbf{X}=-\nabla s\cdot\nabla\psi/(|\nabla\psi|^2)$. The components of the two-contravariant metric tensor g^{ij} are obtained by inverting the matrix $\mathbf{g}_{ij}$.

APPENDIX B: PERTURBED MAGNETIC FIELD IN AXISYMMETRIC MAGNETIC GEOMETRY

Using the magnetic coordinates (ψ,φ,s), the perturbed magnetic vector potential $\delta\mathbf{A}$ is written (in the gauge $\delta\mathbf{A}\cdot\nabla\varphi\equiv 0$) as

$$\delta\mathbf{A}=\delta A_\parallel\nabla s-\delta\beta\nabla\psi\equiv\nabla\,\delta\alpha+\delta\psi\nabla\varphi-\delta\chi\nabla\psi, \tag{B1}$$

where $\delta A_\parallel\equiv\partial_s\delta\alpha$, $\delta\psi\equiv-\partial_\varphi\delta\alpha$, and $\delta\chi\equiv\delta\beta+\partial_\psi\delta\alpha$. Hence, the perturbed magnetic field $\delta\mathbf{B}\equiv\nabla\times\delta\mathbf{A}$ is written as

$$\delta\mathbf{B}=\nabla\,\delta\psi\times\nabla\varphi+\nabla\psi\times\nabla\,\delta\chi. \tag{B2}$$

Since magnetic fields are most easily measured in terms of magnetic fluxes across surfaces S (i.e., the perturbed magnetic flux through a constant-ψ^k surface is

$$\delta\Psi^k\equiv\int_S\frac{\delta B^k}{B}\,\epsilon_{ijk}\,d\psi^i\,d\psi^j,$$

where $\epsilon_{ijk}B^{-1}\,d\psi^i d\psi^j$ denotes the infinitesimal surface element on the constant-ψ^k surface), the perturbed magnetic field Eq. (B2) has the following contravariant components $\delta B^i\equiv\delta\mathbf{B}\cdot\nabla\psi^i$:

$$\begin{aligned}\delta B^\psi&=-B\,\partial_s\delta\psi\equiv B\,\partial_s(\partial_\varphi\delta\alpha),\\ \delta B^\varphi&=-B\,\partial_s\delta\chi\equiv-B\,\partial_s(\delta\beta+\partial_\psi\delta\alpha),\\ \delta B^s&=B(\partial_\psi\delta\psi+\partial_\varphi\delta\chi)\equiv B\,\partial_\varphi\delta\beta.\end{aligned} \tag{B3}$$

Note that the perturbed magnetic flux through a constant-ψ surface is $\delta\Psi^\psi=\delta\alpha$ and that the parallel component of the perturbed magnetic field $\delta B_\parallel\equiv\hat{\mathbf{b}}\cdot\delta\mathbf{B}$ is written in terms of these components as $\delta B_\parallel=\delta B^s+a\,\delta B^\psi=B\,\partial_\varphi(\delta\beta+a\,\partial_s\delta\alpha)$.

The perturbed electric field is written in terms of the perturbed scalar potential $\delta\phi$ and the perturbed vector potential Eq. (B1) as

$$\delta\mathbf{E}=-\nabla\,\delta\phi-c^{-1}(\partial_t\delta A_\parallel\nabla s-\partial_t\delta\beta\nabla\psi). \tag{B4}$$

Since electric fields are most easily measured in terms of voltages along curves C (i.e., the perturbed loop voltage along loop C is $\int_C - \delta E_i \, d\psi^i$), the perturbed electric field Eq. (B4) has the following covariant components $\delta E_i \equiv \delta \mathbf{E} \cdot \partial_i \mathbf{X}$:

$$\begin{aligned} \delta E_\psi &= -\partial_\psi \delta\phi + c^{-1} \partial_t \delta\beta, \\ \delta E_\varphi &= -\partial_\varphi \delta\phi, \\ \delta E_\parallel &= -\partial_s \delta\Phi \equiv -\partial_s (\delta\phi + c^{-1} \partial_t \delta\alpha). \end{aligned} \tag{B5}$$

Lastly, we note that the perturbation Hamiltonian equation (49)

$$\delta\tilde{K}_m \equiv q\,\delta\tilde{\Phi}_m + J_g \omega_g \frac{\delta\tilde{B}_{\parallel m}}{B} + \frac{imq}{c}\,\omega_d\,\delta\tilde{\alpha}_m$$

can now be expressed in terms of the perturbed magnetic and electric fields, with

$$\delta\tilde{\Phi}_m \equiv -\int \delta\tilde{E}_{\parallel m}\, ds,$$

$$\delta\tilde{\alpha}_m \equiv -\frac{i}{m}\int \frac{\delta\tilde{B}_m^\psi}{B} ds = \frac{c}{m\omega_\kappa}\delta\tilde{E}_{\varphi m} - \frac{ic}{\omega_\kappa}\int \delta\tilde{E}_{\parallel m}\, ds,$$

where the second expression for $\delta\tilde{\alpha}_m$ is based on the identity

$$\delta\tilde{E}_{\parallel m} \equiv \frac{-i}{m}\frac{\partial \delta\tilde{E}_{\varphi m}}{\partial s} + \frac{\omega_\kappa}{mc}\frac{\delta\tilde{B}_m^\psi}{B}.$$

Thus we obtain two useful expressions for the perturbation Hamiltonian

$$\delta\tilde{K}_m = -q\left(\int \delta\tilde{E}_{\parallel m}\, ds - \frac{\omega_d}{c}\int \frac{\delta\tilde{B}_m^\psi}{B} ds\right) + J_g\omega_g \frac{\delta\tilde{B}_{\parallel m}}{B} \tag{B6}$$

$$= \frac{q}{mc}(m\omega_d - \omega_\kappa)\int \frac{\delta\tilde{B}_m^\psi}{B} ds + J_g\omega_g\frac{\delta\tilde{B}_{\parallel m}}{B} + \frac{iq}{m}\delta\tilde{E}_{\varphi m}. \tag{B7}$$

Using Eq. (B6), we obtain the following expression for the perturbed parallel force:

$$\delta\dot{p}_{\parallel m} = -\frac{\partial \delta\tilde{K}_m}{\partial s} = \hat{\mathbf{b}} \cdot \left(q\,\delta\tilde{\mathbf{E}}_m + \frac{q\dot{\mathbf{X}}_0}{c} \times \delta\tilde{\mathbf{B}}_m \right) - \frac{J_g\omega_g}{B}\frac{\partial \delta\tilde{B}_{\parallel m}}{\partial s},$$

in which the last term represents the standard perturbed mirror force and where we have used $\hat{\mathbf{b}} \cdot (\dot{\mathbf{X}}_0 \times \delta\mathbf{B}) = -\omega_d\,\delta B^\psi/B$. Using Eq. (B7), on the other hand, we find that the first term can be written as $-i(\omega_\kappa - m\omega_d)\delta J_d$, where $\delta J_d \equiv (q/c)\,\delta\alpha$ is the perturbed drift action.

Finally, although we assume the stationary magnetic field $\mathbf{B}$ is axisymmetric, deviations from axisymmetry can be incorporated by adding a zero-frequency magnetic perturbation $\delta\mathbf{B_a}$, in addition to the time-dependent (fluctuation) fields. For example, the day–night asymmetry of the magnetosphere can be accounted for by adding an $m=1$ perturbation to the axisymmetric field.

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[20] In terms of magnetic coordinates $\psi^i = (\psi, \varphi, s)$, the divergence of an arbitrary vector field $\mathbf{G}$ is written as $\nabla \cdot \mathbf{G} \equiv B\,\partial_i(B^{-1} G^i)$, where $G^i \equiv \mathbf{G} \cdot \nabla\psi^i$ and $\partial_i \equiv \partial/\partial\psi^i$.
[21] We note that if we were to use the bounce-action $J_b \equiv (2\pi)^{-1}\oint p_\parallel\, ds$ instead of the energy variable ε, the factor τ_b appearing in Eq. (38) would disappear since $\partial J_b/\partial\varepsilon \equiv \tau_b/2\pi$.
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