RADIATION BELT
TRANSPORT THEORY USING
PHASE-SPACE LAGRANGIAN METHODS

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• Outstanding problem in Magnetospheric Physics:

**Transport and Energization of Radiation-Belt Particles during Magnetic Storms**

• During magnetic storms:
  
  ○ MeV electron fluxes in Earth’s magnetosphere typically drop by a few orders of magnitude early in the storm.

  ○ Fluxes then rise to values one to two orders of magnitude above pre-storm values over a period of a day or two.
• Recent observational studies have shown strong correlations between these flux variations and the occurrence of hydromagnetic fluctuations in the 2 to 10 mHz (ULF) frequency range.

• For a 1-MeV test electron at geosynchronous orbit:
  
  o Cyclotron frequency in the range 1 – 3 kHz
  o Bounce frequency in the range 1 – 3 Hz
  o Drift frequency for equatorially-trapped electrons in the range of a few mHz.

• Simulation results by Elkington, Hudson, and Chan (1999) show enhanced MeV electron fluxes during MHD wave activity in ULF range.

![MeV Trapped Electrons can easily encounter Magnetic-Drift Resonances with ULF Waves](Stochastic Transport)
We present a First-Principles Derivation of a Reduced Relativistic Quasilinear Diffusion Equation in Axisymmetric Magnetic Geometry

- Stochastic transport due to low-frequency hydromagnetic fluctuations which conserve first adiabatic invariant

- Formalism allows for
  - bounce resonances
  - drift-bounce resonances
  - drift-resonances
• Relativistic Drift-Kinetic Vlasov Equation

\[ \frac{\partial F}{\partial t} + \left( \frac{B^*}{B^*_{\parallel}} \frac{\partial H}{\partial p_{\parallel}} + \frac{c\hat{b}}{qB^*_{\parallel}} \times \nabla H \right) \cdot \nabla F - \frac{B^*_{\parallel}}{B^*_{\parallel}} \cdot \nabla H \frac{\partial F}{\partial p_{\parallel}} = 0 \]

where \( F \equiv \) relativistic electron drift-kinetic Vlasov distribution

\[ B^* = B + \frac{cp_{\parallel}}{q} \nabla \times \hat{b} \quad \text{and} \quad B^*_{\parallel} = \hat{b} \cdot B^* \]

• Relativistic Drift-Kinetic Hamiltonian

\[ H = \frac{\mathcal{E}}{\gamma - 1} Mc^2 + \epsilon q \left( \delta \phi - \frac{qv_{\parallel}}{c} \delta A_{\parallel} \right) + \epsilon J_g \omega_g \frac{\delta B_{\parallel}}{B} \equiv \mathcal{E} + \epsilon \delta H \]
• Drift-Kinetic Equation in \((X, \mathcal{E})\)-Space

\[
\left( \frac{\partial}{\partial t} + \dot{X}_0 \cdot \nabla \right) F = \epsilon \left[ \frac{c\hat{b}}{qB^*_\parallel} \cdot \nabla F \times \nabla \delta H + \dot{X}_0 \cdot \left( \nabla \delta H \frac{\partial F}{\partial \mathcal{E}} - \nabla F \frac{\partial \delta H}{\partial \mathcal{E}} \right) \right]
\]

where the unperturbed relativistic GC velocity is

\[
\dot{X}_0 = \left( \frac{p_\parallel}{\gamma M} \right) \hat{b} + \frac{c\hat{b}}{\gamma qB^*_\parallel} \times \left( J_g \nabla \omega_g + \frac{p_\parallel^2}{M} \hat{b} \cdot \nabla \hat{b} \right)
\]

• Jacobian Identities \( \mathcal{D} \equiv B^*_\parallel / |v_\parallel| \)

\[
\nabla \cdot (\dot{X}_0 \mathcal{D}) \equiv 0 \quad \text{and} \quad \nabla \times \left( \frac{c\mathcal{D}\hat{b}}{qB^*_\parallel} \right) \equiv \frac{\partial}{\partial \mathcal{E}} (\dot{X}_0 \mathcal{D})
\]
QUASILINEAR ANALYSIS

- **Quasilinear Decomposition**

\[
F = F_0(I; \tau = \epsilon^2 t) + \epsilon \delta F
\]

○ Introduce

\[
\langle \cdots \rangle \equiv \begin{cases} 
\text{fast-time-scale averaging} \\
\text{and} \\
\text{azimuthal-angle averaging}
\end{cases}
\]

- **Lowest-Order Slow-Time-Scale Evolution**

\[
\dot{X}_0 \cdot \nabla F_0 = 0
\]

- **Second-Order Slow-Time-Scale Evolution**

\[
\frac{\partial F_0}{\partial \tau} = \left[ \dot{X}_0 \cdot \left( \nabla \delta H \frac{\partial \delta F}{\partial \mathcal{E}} - \nabla \delta F \frac{\partial \delta H}{\partial \mathcal{E}} \right) \right] \\
+ \frac{c \hat{b}}{q B^*} \cdot \left( \nabla \delta F \times \nabla \delta H \right)
\]
• Fast-Time-Scale (Order $\epsilon$) Dynamics

  ○ Adiabatic-Nonadiabatic Decomposition

  \[ \delta F \equiv \delta H \frac{\partial F_0}{\partial \mathcal{E}} + \delta G \]

  ○ Linearized Nonadiabatic Drift-Kinetic Equation

  \[ \hat{\mathcal{L}} \delta G = i \hat{F} \delta H \]

  where

  \[ \hat{\mathcal{L}} \equiv \frac{\partial}{\partial t} + \dot{X}_0 \cdot \nabla \]

  \[ \hat{F} \equiv i \left( \frac{\partial F_0}{\partial \mathcal{E}} \frac{\partial}{\partial t} - \frac{c\hat{b}}{qB^*_\parallel} \cdot \nabla F_0 \times \nabla \right) \]
AXISYMMETRIC MAGNETIC GEOMETRY

- Magnetic Coordinates \((\psi, \varphi, s)\)

\[
\begin{align*}
B & \equiv \nabla \psi \times \nabla \varphi \\
\hat{b} & \equiv \frac{\partial X}{\partial s} = \nabla s + a(\psi, s) \nabla \psi
\end{align*}
\]

\[
\nabla \times \hat{b} = \hat{b} \times \left( \frac{\partial a}{\partial s} \nabla \psi \right)
\]

\[
B^* = B \hat{b} + \left( p\| \frac{\partial a}{\partial s} \right) \frac{c\hat{b}}{q} \times \nabla \psi
\]

\[
B^*_\parallel = B \rightarrow D = \frac{B}{|v\|}
\]
• Unperturbed Operators

\[ \hat{\mathcal{L}} = \frac{\partial}{\partial t} + v_\parallel \frac{\partial}{\partial s} + \omega_d \frac{\partial}{\partial \varphi} \]

\[ \hat{\mathcal{F}} = i \left( \frac{\partial F_0}{\partial \mathcal{E}} \frac{\partial}{\partial t} - \frac{c}{q} \frac{\partial F_0}{\partial \psi} \frac{\partial}{\partial \varphi} \right) \]

where the drift frequency \( \omega_d(s; \mathcal{E}, \psi, J_g) \) is

\[ \omega_d \equiv \frac{c}{q\gamma} J_g \left[ \frac{\partial \omega_g}{\partial \psi} - a \frac{\partial \omega_g}{\partial s} \right] + \frac{p_\parallel^2}{M} \left( \frac{\partial a}{\partial s} \right) \]

• Drift-Action Invariant \( J_d \)

\[ \dot{X}_0 \cdot \nabla \psi = 0 \implies J_d \equiv \frac{q}{c} \psi \]
• Perturbed Magnetic Field

○ Perturbed vector potential

\[ \delta A \equiv \nabla \delta \alpha + \delta \psi \nabla \varphi - \delta \chi \nabla \psi \]

\[ \delta A_{\parallel} = \frac{\partial \delta \alpha}{\partial s} \]

○ Perturbed magnetic field

\[ \delta B \equiv \nabla \delta \psi \times \nabla \varphi + \nabla \psi \times \nabla \delta \chi \]

• Perturbed Electric Field

\[ \delta E = -\nabla \delta \Phi - \frac{1}{c} \left( \frac{\partial \delta \psi}{\partial t} \nabla \varphi - \frac{\partial \delta \chi}{\partial t} \nabla \psi \right) \]

\[ \delta E_{\parallel} = -\frac{\partial \delta \Phi}{\partial s} \]

where \( \delta \Phi \equiv \delta \phi + c^{-1} \partial_t \delta \alpha \).
• Fourier Decomposition

\[
\begin{pmatrix}
\delta G \\
\delta H
\end{pmatrix}
\equiv \sum_{\kappa} \sum_{m=-\infty}^{\infty} \begin{pmatrix}
\delta \tilde{G}_m(s, I, \omega_\kappa) \\
\delta \tilde{H}_m(s, I, \omega_\kappa)
\end{pmatrix} e^{im\varphi - \omega_\kappa t}
\]

where \( I = (J_g, E, J_d) \).

- Unperturbed Operators

\[
\hat{\mathcal{L}} \rightarrow \mathcal{L} = \sigma |v|| \frac{\partial}{\partial s} - i [\omega_\kappa - m \omega_d(s)]
\]

\[
\hat{\mathcal{F}} \rightarrow \mathcal{F} = \omega_\kappa \frac{\partial F_0}{\partial \mathcal{E}} + \frac{mc}{q} \frac{\partial F_0}{\partial \psi}
\]

- Fast-Time-Scale Averaging

\[
\overline{fg} \equiv \sum_{m,\kappa} \tilde{f}^*_m(\psi, s; \omega_\kappa) \tilde{g}_m(\psi, s; \omega_\kappa)
\]
Bounce-Time-Scale Averaging

\[ \langle f \rangle (I) \equiv \frac{1}{\tau_b} \sum_{\sigma = \pm 1} \int_{s_L}^{s_U} \frac{ds}{\|v\|} f(s, \sigma; I) \]

with \((s_L, s_U) \equiv \text{bounce points along a field line and}\)

\[ \tau_b(I) \equiv \sum_{\sigma} \int_{s_L}^{s_U} \frac{\mathcal{D} ds}{B} = 2 \int_{s_L}^{s_U} \frac{ds}{\|v\|} \equiv 2\pi \frac{\partial J_b}{\partial \mathcal{E}} \]

Second-order Slow-Time Evolution

\[ \frac{\partial F_0}{\partial \tau} \equiv \frac{1}{\tau_b} \frac{\partial}{\partial \psi} \left[ \tau_b \left\langle \frac{\hat{b}}{qB} \times \nabla \psi \cdot (\delta G \nabla \delta H) \right\rangle \right] 
+ \frac{1}{\tau_b} \frac{\partial}{\partial \mathcal{E}} \left[ \tau_b \left\langle \dot{X}_0 \cdot (\delta G \nabla \delta H) \right\rangle \right] \]

where

\[ \left\langle \dot{X}_0 \cdot (\delta G \nabla \delta H) \right\rangle \equiv \sum_{m, \kappa} \omega_\kappa \text{Im} \left\langle \delta \tilde{G}_m \delta \tilde{H}^*_m \right\rangle \]

\[ \left\langle \frac{\hat{b}}{B} \times \nabla \psi \cdot (\delta G \nabla \delta H) \right\rangle \equiv \sum_{m, \kappa} m \text{Im} \left\langle \delta \tilde{G}_m \delta \tilde{H}^*_m \right\rangle \]
Relativistic Quasilinear Drift-Kinetic Diffusion Equation

\[
\frac{\partial F_0}{\partial \tau} \equiv \frac{1}{\tau_b} \frac{\partial}{\partial \mathcal{E}} \left[ \tau_b \left( D_{\mathcal{E}}\mathcal{E}_{QL} \frac{\partial F_0}{\partial \mathcal{E}} + D_{\mathcal{E}}\psi_{QL} \frac{\partial F_0}{\partial \psi} \right) \right] + \frac{1}{\tau_b} \frac{\partial}{\partial \psi} \left[ \tau_b \left( D_{\psi}\mathcal{E}_{QL} \frac{\partial F_0}{\partial \mathcal{E}} + D_{\psi}\psi_{QL} \frac{\partial F_0}{\partial \psi} \right) \right]
\]

where the relativistic quasilinear bounce-averaged diffusion coefficients are

\[
D_{\mathcal{E}}\mathcal{E}_{QL}(I) \equiv \sum_{m,\kappa} \omega_\kappa^2 \hat{\Gamma}_m(I; \omega_\kappa)
\]

\[
D_{\psi}\mathcal{E}_{QL}(I) \equiv \sum_{m,\kappa} (mc/q) \omega_\kappa \hat{\Gamma}_m(I; \omega_\kappa) \equiv D_{\mathcal{E}}\psi_{QL}(I)
\]

\[
D_{\psi}\psi_{QL}(I) \equiv \sum_{m,\kappa} (mc/q)^2 \hat{\Gamma}_m(I; \omega_\kappa)
\]

with

\[
\hat{\Gamma}_m(I; \omega_\kappa) \equiv \mathcal{F}^{-1} \text{Im} \left\langle \delta\tilde{G}_m \delta\tilde{H}_{m}^* \right\rangle
\]
QUASILINEAR POTENTIAL $\hat{\Gamma}_m(I; \omega_\kappa)$

- Linear Nonadiabatic Drift-Kinetic Equation

$$\mathcal{L} \delta \tilde{G}'_m(\sigma, s) \equiv i\mathcal{F} \delta \tilde{K}_m(s)$$

where

$$\delta \tilde{G}'_m \equiv \delta \tilde{G}_m + i \frac{q}{c} \mathcal{F} \delta \tilde{\alpha}_m$$

$$\delta \tilde{K}_m(s) \equiv \delta \tilde{H}_m(\sigma, s) + \mathcal{L} \left( \frac{q}{c} \delta \tilde{\alpha}_m \right)$$

$$= \frac{iq}{m} \delta \tilde{E}_{\varphi m} + J_g\omega_g \frac{\delta \tilde{B}_{\| m}}{B}$$

$$+ \frac{q}{mc} (m\omega_d - \omega_\kappa) \int \frac{\delta \tilde{B}_m^\psi}{B} \, ds$$

We can show that

$$\hat{\Gamma}_m = \mathcal{F}^{-1} \text{Im} \left\langle \delta \tilde{G}'_m \delta \tilde{K}^*_m \right\rangle$$
• General Solution for $\delta \tilde{G}'_m(\sigma, s)$

$$F^{-1} \delta \tilde{G}'_m(\sigma, s) =$$

$$i\sigma \ e^{i\sigma \theta(s)} \left[ \int_{s_L}^{s} \frac{ds'}{|v|} \ e^{-i\sigma \theta(s')} \ \delta \tilde{K}_m(s') \right]$$

$$- \ \frac{\tau_b}{2} \ e^{i\sigma \theta(s)} \left[ \cot \Theta \ \left< \delta \tilde{K}_m \ \cos \theta \right> + \left< \delta \tilde{K}_m \ \sin \theta \right> \right]$$

where

$$\theta(s) = \int_{s_L}^{s} \frac{ds'}{|v|} \ \left[ \omega_\kappa - m \omega_d(s') \right]$$

$$\Theta = \int_{s_L}^{s_U} \frac{ds}{|v|} \ \left[ \omega_\kappa - m \omega_d(s) \right]$$

$$\equiv \ \frac{\tau_b}{2} \ (\omega_\kappa - m \left< \omega_d \right>)$$

○ Note: $\delta \tilde{G}'_m(\sigma, s)$ satisfies matching conditions

$$\delta \tilde{G}'_m(\sigma = +1, \ s = s_L) = \delta \tilde{G}'_m(\sigma = -1, \ s = s_L)$$

$$\delta \tilde{G}'_m(\sigma = +1, \ s = s_U) = \delta \tilde{G}'_m(\sigma = -1, \ s = s_U)$$
• Quasilinear perturbation Potential

\[ \hat{\Gamma}_m \equiv \frac{T_b}{2} \left| \langle \delta \tilde{K}_m \cos \theta \rangle \right|^2 \text{Im}(- \cot \Theta) \]

where

\[ \cot \Theta \equiv \frac{1}{\pi} \sum_{n=\pm \infty}^{\infty} \frac{\omega_b}{(\omega - m \langle \omega_d \rangle - n \omega_b)} \]

\[ \text{Im}(- \cot \Theta) \equiv \sum_{n=\pm \infty}^{\infty} \omega_b \frac{\delta (\omega - m \langle \omega_d \rangle - n \omega_b)}{\text{wave–particle resonances}} \]

\[ \hat{\Gamma}_m(I; \omega_\kappa) \equiv \sum_{n=\pm \infty}^{\infty} \pi \delta (\omega - m \langle \omega_d \rangle - n \omega_b) \]

\[ \times \left| \langle \delta \tilde{K}_m(s; I, \omega_\kappa) \cos \theta(s; I, \omega_\kappa) \rangle \right|^2 \]
• Relativistic Quasilinear Diffusion Tensor

\[ D_{\text{QL}}^{\varepsilon\varepsilon} \equiv \sum_{m,n,\kappa} \omega_{\kappa}^2 \left[ \tau_{ac} \left| \left\langle \delta \tilde{K}_m \cos \theta \right\rangle \right|^2 \right] \]

\[ D_{\text{QL}}^{\psi\varepsilon} \equiv \sum_{m,n,\kappa} \frac{mc\omega_{\kappa}}{q} \left[ \tau_{ac} \left| \left\langle \delta \tilde{K}_m \cos \theta \right\rangle \right|^2 \right] \equiv D_{\text{QL}}^{\varepsilon\psi} \]

\[ D_{\text{QL}}^{\psi\psi} \equiv \sum_{m,n,\kappa} \left( \frac{mc}{q} \right)^2 \left[ \tau_{ac} \left| \left\langle \delta \tilde{K}_m \cos \theta \right\rangle \right|^2 \right] \]

\[ \tau_{ac} \equiv \pi \delta (\omega_{\kappa} - m \langle \omega_d \rangle - n \omega_b) \]
SUMMARY

• First-Principles Derivation of a Relativistic Quasilinear Drift-Kinetic Diffusion Equation for Applications in Radiation-Belt Transport Theory.

• Relativistic Quasilinear Diffusion Tensor is Bounce-Averaged and contains General Wave Polarizations.

• Present Work generalizes previous work of Chen (1999).

Poster is available in PDF format at http://spacsun.rice.edu/~aac/pubs/APS.pdf