5.1 A relativistic harmonic oscillator

Suppose the relativistic Lagrangian for a particle of mass \(m\) in a one-dimensional harmonic potential is

\[
L = -\frac{mc^2}{\gamma} - V(x),
\]

where \(\gamma = 1/\sqrt{1-\dot{x}^2/c^2}\) and \(V(x) = (1/2)kx^2\). Since \(L\) is independent of time, the total energy \(E = \gamma mc^2 + V\) is a constant of the motion.

(a) Solve \(E = \gamma mc^2 + V\) for \(\dot{x}\) and show that \(a\), the amplitude of the oscillation, is given by

\[
a = \sqrt{\frac{2K_0}{k}},
\]

where \(K_0\) is the maximum kinetic energy of the particle. Note that kinetic energy \(K = (\gamma - 1)mc^2\).

(b) For the case where the potential energy of the particle is always small compared to \(mc^2\), calculate the correction to the nonrelativistic oscillation period to leading order in \(ka^2/2mc^2\).

5.2 Geodesic equations

Consider a system with \(n\) degrees of freedom and generalized coordinates \(q^i(\lambda), i = 1, 2, ..., n\), where \(\lambda\) parametrizes the path of the system through the configuration space.

The action is \(S = \int \sqrt{g_{ij}\dot{q}^i\dot{q}^j} d\lambda\). Assume \(\det(g_{ij}) \neq 0\), so that the inverse matrix \(g^{ij}\) exists \((g^{ij}g_{jk} = \delta^i_k)\).

(a) Show that the Euler-Lagrange equations for this system are

\[
\ddot{q}^i + \Gamma^i_{jk}\dot{q}^j\dot{q}^k = \frac{s}{\dot{s}}\dot{q}^i,
\]

where \(\ddot{q}^i = d^2q^i/d\lambda^2\), \(\dot{s} = ds/d\lambda = \sqrt{g_{ij}\dot{q}^i\dot{q}^j}\), \(\ddot{s} = d^2s/d\lambda^2\), and

\[
\Gamma^i_{jk} = \frac{1}{2}g^{iu}\left(\frac{\partial g_{lj}}{\partial q^k} + \frac{\partial g_{lk}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^l}\right).
\]

(b) What are the Euler-Lagrange equations of part (a) with the choice \(\lambda = s\), the generalized arc length?

Comment: The functions \(g_{ij}\) define a metric on the configuration space, \(\Gamma^i_{jk}\) is called a Christoffel symbol of the second kind, and the equations in part (b) are called the geodesic equations. In general relativity the geodesic equations describe a particle moving in curved spacetime. Lagrangians of the form \(L = \sqrt{g_{ij}\dot{q}^i\dot{q}^j}\) are related to systems called \textit{sigma models}, which appear in condensed matter physics and string theory.

5.3 An illuminating particle decay

A stationary particle of mass \(m\) decays spontaneously into a particle of mass \(m/2\) and a photon. Find the energies, momenta, and speeds of the decay products.

5.4 Read sections 11, 14, 15, and 16 of Landau and Lifshitz.

After you have read these sections, for credit write “I have read sections 11, 14, 15, and 16.”